Assignment 2

Deadline: Jan 25, 2019.

Hand in: Section 6.2 no 9, 10, 18; Supplementary exercise no 1, 5a.

Section 6.2: *Q.5, 9-12, 17, 18, 19.*

Supplementary Exercises

1. Consider the function $f(x) = x|x^2 - 12|$ on $[-2, 3]$. (a) Determine all its local max/min points, (b) all max/min points, (c) subinterval of increasing/decreasing, and (d) sketch its graph.

2. Consider the function $g(x) = x/(x^2 + 1)$ on $(-\infty, \infty)$ and study the same questions as in the previous exercise.

3. Let f be a function defined on $\mathbb{R}$. It is called a periodic function if there is a non-zero number T such that $f(x + T) = f(x)$ for all x. The number T is called a period of f.
   (a) Show that $nT, n \neq 0 \in \mathbb{Z}$, is also a period if f has a period T.
   (b) Let f be differentiable. Show that f must be constant if it has a sequence of periods $\{T_n\}, T_n \to 0$. Hint: If f is non-constant, $f'(c) \neq 0$ at some c.
   (c) (Optional) Let f be differentiable. Show that if f is non-constant, there exists a positive period L satisfying, if T is another period of f, then $T = nL$ for some integer n. This L is called the minimal period of f.

4. Let f be a differentiable function defined on $(0, \infty)$. Suppose f satisfies $|f(x)| \leq C\sqrt{x}$ for all $x \in (0, \infty)$ for some constant $C > 0$. Show that there exists a sequence of numbers $\{x_n\}, x_n \to \infty$, such that $f'(x_n) \to 0$ as $n \to \infty$.

5. (a) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$, where $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Suppose that p has n real roots. Show that p' has $n - 1$ real roots.
   (b) (Optional) What happens when p does not have n real roots? In this case, there are complex roots. Could you make a guess on the distribution of the roots of p'?

6. It has been shown that a differentiable function f on (a, b) satisfying $f'(x) = 0$ everywhere must be a constant. Show that this result is not true when the assumption is relaxed to the right derivative of f exists and $f'_+(x) = 0$ everywhere.